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LETTER TO THE EDITOR

Connection aspects of nonlinear lattice equations in (1+1) dimensions

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Abstract. We consider two nonlinear lattice equations and study some geometrical features. We demonstrate that two nonlinear partial differential equations are deduced from the fact that the SL(2, ℝ) connection has zero curvature.

Partial differential equations (PDES) are usually called integrable if one of the following properties is fulfilled: (1) the initial value problem can be solved exactly with the help of the inverse scattering transform; (2) they have an infinite number of conservation laws; (3) they have an auto Bäcklund transformation or a Bäcklund transformation to a linear equation; (4) besides Lie point vector fields they admit Lie Bäcklund vector fields; (5) they describe pseudospherical surfaces, i.e. surfaces of constant negative Gaussian curvature; (6) they can be written as covariant exterior derivatives of Lie algebra valued differential forms. It is conjectured that if property (1) holds, then properties (2)–(6) also hold.

Furthermore, to test the integrability we can use the Painlevé property. Recently, Ward (1984) has introduced the Painlevé property for PDES. Let n be the number of the independent variables. Assume that the PDE has coefficients which are analytic on C^n. The Painlevé property is defined as follows: if S is an analytic non-characteristic complex hypersurface in C^n, then every solution of the PDE which is analytic on $C^n\backslash S$, is meromorphic on C^n. A weaker form of the Painlevé property given by Ward (1984) was proposed by Weiss *et al* (1983). Examples show that, if a nonlinear equation has the Painlevé property, then this equation is integrable. On the other hand, we cannot conclude, in general, that a PDE which is integrable has the Painlevé property.

Crampin *et al* (1977) have introduced the curvature form on bundles to study some geometrical features of soliton equations in field theory, namely the Korteweg–de Vries (KdV), the modified KdV and the sine-Gordon equations. The basic idea is the relationship between the nonlinear differential equations with soliton solutions and the group SL(2, ℝ). Furthermore, Crampin (1978) has deduced the Bäcklund transformations and the equations for the inverse scattering transform from the fact that the SL(2, ℝ) connection has zero curvature. The connection form is explicitly given for some SL(2, ℝ)-valued functions and it is shown that the connection is 'pure gauge'. In further investigations, I have considered the soliton connection on bundles of the Liouville equation (1981a), the well known nonlinear Schrödinger equation (1981b) and a system of partial differential equations, namely the reduced Maxwell–Bloch equations (1984). This system of nonlinear equations are the governing equations in the theory of optical self-induced transparency and they are important in the physics of nonlinear optics.

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We consider nonlinear lattice equations in the so-called continuum approach. We investigate two lattice equations, namely a lattice which consists of a combined form of the modified KdV equation and the sine-Gordon equation of Konno *et al* (1974)

$$u_{xt}+\tfrac{3}{2}u_x^2u_{xx}+u_{xxxx}-\alpha\sin u=0 \tag{1}$$

and a nonlinear evolution equation which is a combined form of the KdV and modified KdV equation (see Wadati 1975)

$$u_t+6\alpha uu_x+6\beta u^2u_x+u_{xxx}=0. \tag{2}$$

In this letter it is shown that the two lattice equations can be written as covariant exterior derivatives of Lie algebra valued differential forms. The basic idea is to express the curvature form by the covariant exterior derivative of the 1-form ω on a principal bundle $P(M, G)$ with values in a finite-dimensional vector space V. The letter is organised as follows. We give a short presentation of the theory and cite some formulae and then we give an application to the lattice equations. Let us start with the scattering equations

$$\hat{L}\varphi=\varphi_x, \tag{3}$$

where

$$\hat{L}=\begin{pmatrix}\eta & q(x,t)\\ r(x,t) & -\eta\end{pmatrix}. \tag{4}$$

The subscript indicates partial differentiations. φ is a column vector with transpose $\tilde{\varphi}=(\varphi^1,\varphi^2)$. The time evolution of the functions $\varphi^1(x,t)$ and $\varphi^2(x,t)$ is given by

$$\hat{A}\varphi=\varphi_t, \tag{5}$$

where

$$\hat{A}=\begin{pmatrix}A(x,t;\eta) & B(x,t;\eta)\\ C(x,t;\eta) & -A(x,t;\eta)\end{pmatrix}. \tag{6}$$

The parameter η is called the eigenvalue of the scattering problem and the quantities $q(x,t)$, $r(x,t)$, $A(x,t;\eta)$, $B(x,t;\eta)$ and $C(x,t;\eta)$ must be given in order to specify the problem which is under consideration. Rewriting (3) and (4) in matrix notation, we then obtain

$$\frac{\partial\varphi^k}{\partial x^j}+\sum_{p=1}^{2}\Gamma^k_{pj}\varphi^p=0, \tag{7}$$

where $j,k=(1,2)$ and $x^1=x$, $x^2=t$. The quantities $\varphi^q(x,t)$ are interpreted as the components of a two-component field on the principle bundle $P(M,G)$. The quantities Γ^k_{pj} are given by the components of the matrix in (4) and (6).

The curvature form Ω is given by the exterior covariant derivative of the 1-form ω on P with values in a finite-dimensional vector space V. The curvature form can be written

$$\Omega=\nabla\omega=\mathrm{d}\omega\circ h. \tag{8}$$

Ω is a $\mathscr{g}$-valued 2-form and $\nabla\omega(X_1,\ldots,X_{p+1})=\mathrm{d}\omega\,(hX_1,\ldots,hX_{p+1})$, where $h\colon T_p(P(M,G))\to S_p$ is the projection of the tangential space $T_p=S_p\oplus V_p$ onto its horizontal subspace S_p. The space V_p of vertical vectors lies tangential to the fibre.

The exterior derivative d is unchanged in the action on forms which take their values in a real vector space V. On sections of $V \oplus \Lambda^1[T_p(P(M, G))]$ we have

$$\mathrm{d}(\omega^j \otimes X_j) = \mathrm{d}\omega^j \otimes X_j, \qquad \omega^j \in \Lambda^1(T_p), \tag{9}$$

where $\{X_k\}_{k=1}^n$ is a basis of V. If V is a Lie algebra $V = \mathcal{g}$ we can define

$$[\omega^i \otimes X_i, \omega^j \otimes X_j] := (\omega^i \wedge \omega^j) \otimes [X_i, X_j]. \tag{10}$$

Equation (10) relates R-valued forms to the bracket of $\mathcal{g}$-valued forms. The expression (10) is anticommutative and satisfies the Jacobi identity. Now the curvature form (8) can be expressed. Let $\{X_k\}_{k=1}^3$ be a basis of the Lie algebra $\mathcal{g} = \mathrm{SL}(2, \mathbb{R})$ then from (8) with (9) and (10) follows

$$\Omega = \sum_{i=1}^3 \mathrm{d}\omega^i \otimes X_i + \tfrac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 (\omega^i \wedge \omega^j) \otimes [X_i, X_j]. \tag{11}$$

The $\omega^k (k = 1, 2, 3)$ are àrbitrary 1-forms and $[X_p, X_q]$ represents the commutator of the quantities X_k. Let

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{12}$$

be a basis of $\mathcal{g}$. In view of (7) the 1-forms are expressible as follows

$$\omega^1 = -(\eta\, \mathrm{d}x + A\, \mathrm{d}t), \qquad \omega^2 = -(q\, \mathrm{d}x + B\, \mathrm{d}t), \qquad \omega^3 = -(r\, \mathrm{d}x + C\, \mathrm{d}t). \tag{13}$$

If we take into account (12) and (13) then the curvature form (11) is given by

$$\begin{aligned} \Omega = &\ (qC - rB - A_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_1 \\ &+ (2\eta B - 2qA + q_t - B_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_2 \\ &+ (-2\eta C + 2rA + r_t - C_x)\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_3. \end{aligned} \tag{14}$$

(a) Nonlinear one-dimensional lattice equation. With the choice

$$\begin{aligned} r &= -q = \tfrac{1}{2}u, \\ A &= 4\eta^3 - \tfrac{1}{2}\eta u_x^2 + (\alpha/4\eta) \cos u, \\ B &= \tfrac{1}{2}u_{xxx} + \eta u_{xx} + 2\eta^2 u_x + \tfrac{1}{4}u_x^3 + (\alpha/4\eta) \sin u, \\ C &= -\tfrac{1}{2}u_{xxx} + \eta u_{xx} - 2\eta^2 u_x + \tfrac{1}{4}u_x^3 + (\alpha/4\eta) \sin u, \end{aligned} \tag{15}$$

it follows from (14) that

$$\begin{aligned} \Omega = \mathrm{d}x \wedge \mathrm{d}t\{&(-\tfrac{1}{2}u_{xt} - \tfrac{1}{2}u_{xxxx} - \tfrac{3}{4}u_x^2 u_{xx} + \tfrac{1}{2}\alpha \sin u) \otimes X_2 \\ &+ (\tfrac{1}{2}u_{xt} + \tfrac{1}{2}u_{xxxx} + \tfrac{3}{4}u_x^2 u_{xx} - \tfrac{1}{2}\alpha \sin u) \otimes X_3\} \end{aligned} \tag{16}$$

or

$$\Omega = (u_{xt} + \tfrac{3}{2}u_x^2 u_{xx} + u_{xxxx} - \alpha \sin u)\, \mathrm{d}x \wedge \mathrm{d}t \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

If $\Omega = 0$ then it follows that

$$u_{xt} + \tfrac{3}{2}u_x^2 u_{xx} + u_{xxxx} - \alpha \sin u = 0. \tag{18}$$

(b) If we choose

$$\begin{aligned} &q = u, \qquad r = -\alpha - \beta u, \\ &A = -4\eta^3 + 2\eta q r + r q_x - q r_x, \\ &B = -4\eta^2 q - 2\eta q_x + 2q^2 r - q_{xx}, \\ &C = -4\eta^2 r + 2\eta r_x + 2q r^2 - r_{xx}, \end{aligned} \tag{19}$$

it follows from (14) that

$$\begin{aligned} \Omega = {} & (u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx})\, \mathrm{d}x \wedge \mathrm{d}t \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ & - \beta(u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx})\, \mathrm{d}x \wedge \mathrm{d}t \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{20}$$

From $\Omega = 0$ we conclude that

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0. \tag{21}$$

Concluding remarks. It is pointed out that the linear scattering problem for the lattice equations may be described in terms of a linear connection on a principal $SL(2, \mathbb{R})$-bundle. From the condition $\Omega = 0$, we conclude that ω satisfies the structure equation of Maurer-Cartan, $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0$, and that the connection in $P(M, G)$ is flat. Consequently the flat connection has zero curvature. Therefore, the lattice equations are integrable.

A valuable test for integrability of PDEs is the Painlevé property. Let us consider here the weaker form of the Painlevé property of Weiss *et al* (1983). The solution of a given PDE can be represented locally as a single-valued expansion about its movable singular manifold. This means that if u is a solution of a PDE, we can write the Painlevé expansion

$$u = \Phi^n \sum_{j=0}^{\infty} u_j \Phi^j, \tag{22}$$

where Φ is the analytic function for which the equation $\Phi = 0$ defines the singular manifold. We cannot apply the Painlevé test directly to equation (1) because of the nonlinearity of the term $\sin u$. Therefore we introduce the transformation $v = \exp(\mathrm{i}u)$ into equation (1) and obtain

$$2v^3 v_{xxxx} - 8v^2 v_x v_{xxx} - 6v^2 v_{xx}^2 + 21 v v_x^2 v_{xx} - 9v_x^4 - 2v^2 v_x v_t + 2v^3 v_{tx} - \alpha v^5 + \alpha v^3 = 0. \tag{23}$$

If we perform the Painlevé test we obtain for the dominant behaviour $n = -4$ and the resonances at $r_1 = -3$, $r_2 = -1$, $r_3 = 4$ and $r_4 = 6$ for equation (23), and $n = -1$ and the resonances at $r_1 = -1$, $r_2 = 3$ and $r_3 = 4$ for equation (2). Moreover, we obtain that both nonlinear equations are integrable in the sense of Weiss.

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